

SOME HERMITE-HADAMARD-FEJER TYPE INEQUALITIES FOR HARMONICALLY CONVEX FUNCTIONS VIA FRACTIONAL INTEGRAL

İMDAT İŞCAN, SERCAN TURHAN, AND SELAHATTİN MADEN

ABSTRACT. In this paper, we gave the new general identity for differentiable functions. As a result of this identity some new and general inequalities for differentiable harmonically-convex functions are obtained.

1. INTRODUCTION

The classical or the usual convexity is defined as follows:

A function $f : I \longrightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

A number of papers have been written on inequalities using the classical convexity and one of the most captivating inequalities in mathematical analysis is stated as follows

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

where $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a \leq b$. Both the inequalities hold in reversed direction if f is concave. The inequalities stated in (1.1) are known as Hermite-Hadamard inequalities.

For more results on (1.1) which provide new proof, significantly extensions, generalizations, refinements, counterparts, new Hermite-Hadamard-type inequalities and numerous applications, we refer the interested reader to [2, 3, 5, 6, 8, 9, 12, 13, 15, 16] and the references therein.

The usual notion of convex function have been generalized in diverse manners. One of them is the so called harmonically s-convex functions and is stated in the definition below.

Definition 1. [5, 7] Let $I \subset (0, \infty)$ be a real interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be harmonically s-convex (concave), if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (\geq) t^s f(y) + (1-t)^s f(x)$$

holds for all $x, y \in I$ and $t \in [0, 1]$, and for some fixed $s \in (0, 1]$.

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It can be easily seen that for $s = 1$ in Definition 1 reduces to following Definition 2:

Definition 2. [6] A function $f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is said to be harmonically-convex function, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If the inequality is reversed, then f is said to be harmonically concave.

Proposition 1. [6] Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval and $f : I \rightarrow \mathbb{R}$ is function, then:

if $I \subset (0, \infty)$ and f is convex and nondecreasing function then f is harmonically convex.

if $I \subset (0, \infty)$ and f is harmonically convex and nonincreasing function then f is convex.

if $I \subset (-\infty, 0)$ and f is harmonically convex and nondecreasing function then f is convex.

if $I \subset (-\infty, 0)$ and f is convex and nonincreasing function then f is harmonically convex.

For the properties of harmonically-convex functions and harmonically-s-convex function, we refer the reader to [1, 5, 6, 7, 8, 10, 11] and the reference there in.

Most recently, a number of findings have been seen on Hermite-Hadamard type integral inequalities for harmonically-convex and for harmonically-s-convex functions.

In [14], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be convex function. Then the inequality

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x)dx,$$

holds, where $g : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$.

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2) see [15].

In [6], İşcan gave definition of harmonically convex functions and established following Hermite-Hadamard type inequality for harmonically convex functions as follows:

Theorem 2. [15] Let $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ then the following inequalities hold:

$$(1.3) \quad f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a)+f(b)}{2}.$$

In [11], İşcan and Wu represented Hermite-Hadamard's inequalities for harmonically convex functions in fractional integral form as follows:

Theorem 3. [11] Let $f : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If f is harmonically-convex on $[a, b]$, then the following inequalities for fractional integrals hold:

$$(1.4) \quad f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a}\right)^\alpha \left\{ \begin{array}{l} J_{1/a^-}^\alpha (f \circ h)(1/b) \\ + J_{1/b^+}^\alpha (f \circ h)(1/a) \end{array} \right\} \\ \leq \frac{f(a) + f(b)}{2}.$$

with $\alpha > 0$ and $h(x) = 1/x$.

Definition 3. A function $g : [a, b] \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is said to be harmonically symmetric with respect to $2ab/a + b$ if

$$(1.5) \quad g(x) = g\left(\frac{1}{\frac{1}{a} + \frac{1}{b} - \frac{1}{x}}\right)$$

holds for all $x \in [a, b]$.

In [1] Chan and Wu represented Hermite-Hadamard-Fejer inequality for harmonically convex functions as follows:

Theorem 4. Suppose that $f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be harmonically-convex function and $a, b \in I$, with $a < b$. If $f \in L[a, b]$ and $g : [a, b] \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/a + b$, then

$$(1.6) \quad f\left(\frac{2ab}{a+b}\right) \int_a^b \frac{g(x)}{x^2} dx \leq \int_a^b \frac{f(x)g(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x^2} dx$$

In [10] İşcan and Kunt represented Hermite-Hadamard-Fejer type inequality for harmonically convex functions in fractional integral forms and established following identity as follows:

Theorem 5. Let $f : [a, b] \rightarrow \mathbb{R}$ be harmonically convex function with $a < b$ and $f \in L[a, b]$. If $g : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/a + b$, then the following inequalities for fractional integrals hold:

$$(1.7) \quad f\left(\frac{2ab}{a+b}\right) \left[J_{1/a^-}^\alpha (g \circ h)(1/b) + J_{1/b^+}^\alpha (g \circ h)(1/a) \right] \\ \leq \left[J_{1/a^-}^\alpha (fg \circ h)(1/b) + J_{1/b^+}^\alpha (fg \circ h)(1/a) \right] \\ \leq \frac{f(a) + f(b)}{2} \left[J_{1/a^-}^\alpha (g \circ h)(1/b) + J_{1/b^+}^\alpha (g \circ h)(1/a) \right]$$

with $\alpha > 0$ and $h(x) = 1/x$, $x \in [\frac{1}{b}, \frac{1}{a}]$.

Definition 4. Let $f \in L[a, b]$. The right-hand side and left-hand side Hadamard fractional integrals $J_{a^+}^\alpha f$ and $J_{b^-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1}$ and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$

Lemma 1. For $0 < \theta \leq 1$ and $0 < a \leq b$ we have

$$|a^{\theta} - b^{\theta}| \leq (b-a)^{\theta}.$$

In [4] D. Y. Hwang found out a new identity and by using this identity, established a new inequalities. Then in [12] İ. İşcan and S. Turhan used this identity for GA-convex functions and obtain generalized new inequalities. In this paper, we established a new inequality similar to inequality in [12] and then we obtained some new and general integral inequalities for differentiable harmonically-convex functions using this lemma. The following sections, let the notion, $L(t) = \frac{aH}{tH+(1-t)a}$, $U(t) = \frac{bH}{tH+(1-t)b}$ and $H = H(a, b) = \frac{2ab}{a+b}$.

2. MAIN RESULT

Throughout this section, let $\|g\|_{\infty} = \sup_{t \in [a, b]} |g(x)|$, for the continuous function $g : [a, b] \rightarrow [0, \infty)$ be differentiable mapping I^o , where $a, b \in I$ with $a \leq b$, and $h : [a, b] \rightarrow [0, \infty)$ be differentiable mapping.

Lemma 2. If $f' \in L[a, b]$ then the following inequality holds:

$$(2.1) \quad \begin{aligned} & [h(b) - 2h(a)] \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \int_a^b f(x) h'(x) dx \\ &= \frac{b-a}{4ab} \left\{ \int_0^1 [2h(L(t)) - h(b)] f'(L(t)) (L(t))^2 dt \right. \\ & \quad \left. + \int_0^1 [2h(U(t)) - h(b)] f'(U(t)) (U(t))^2 dt \right\} \end{aligned}$$

Proof. By the integration by parts, we have

$$\begin{aligned} I_1 &= \int_0^1 [2h(L(t)) - h(b)] d(f(L(t))) \\ &= [2h(L(t)) - h(b)] f(L(t)) \Big|_0^1 \\ &\quad - \left(\frac{1}{a} - \frac{1}{b} \right) \int_0^1 f(L(t)) h'(L(t)) (L(t))^2 dt \end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_0^1 [2h(U(t)) - h(b)] d(f(U(t))) \\
&= [2h(U(t)) - h(b)] f(U(t)) \Big|_0^1 \\
&\quad - \left(\frac{1}{a} - \frac{1}{b} \right) \int_0^1 f(U(t)) h'(U(t)) (U(t))^2 dt
\end{aligned}$$

Therefore

$$\begin{aligned}
(2.2) \quad \frac{I_1 + I_2}{2} &= [h(b) - 2h(a)] \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \frac{b-a}{2ab} \left\{ \int_0^1 f(L(t)) h'(L(t)) (L(t))^2 dt \right. \\
&\quad \left. + \int_0^1 f(U(t)) h'(U(t)) (U(t))^2 dt \right\}
\end{aligned}$$

This complete the proof \square

Lemma 3. For $a, H, b > 0$, we have

$$(2.3) \quad \zeta_1(a, b) = \int_0^1 |2h(L(t)) - h(b)| (1-t) (L(t))^2 dt$$

$$(2.4) \quad \zeta_2(a, b) = \int_0^1 t (L(t))^2 |2h(L(t)) - h(b)| dt + \int_0^1 t ((U(t))^2 |2h(U(t)) - h(b)| dt$$

$$(2.5) \quad \zeta_3(a, b) = \int_0^1 |2h(U(t)) - h(b)| (1-t) (U(t))^2 dt$$

Theorem 6. Let $f : I \subseteq \mathbb{R} = (0, \infty) \longrightarrow \mathbb{R}$ be differentiable mapping I° , where $a, b \in I$ with $a < b$. If the mapping $|f'|$ is harmonically-convex on $[a, b]$, then the following inequality holds:

$$\begin{aligned}
(2.6) \quad &\left| [h(b) - 2h(a)] \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \int_a^b f(x) h'(x) dx \right| \\
&\leq \frac{b-a}{4ab} [\zeta_1(a, b) |f'(a)| + \zeta_2(a, b) |f'(H)| + \zeta_3(a, b) |f'(b)|]
\end{aligned}$$

where $\zeta_1(a, b), \zeta_2(a, b), \zeta_3(a, b)$ are defined in Lemma 3.

Proof. Continuing equality (2.1) in Lemma 2

$$(2.7) \quad \left| [h(b) - 2h(a)] \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \int_a^b f(x)h'(x)dx \right|$$

$$\leq \frac{b-a}{4ab} \left\{ \int_0^1 |2h(L(t)) - h(b)| |f'(L(t)) (L(t))^2| dt \right.$$

$$\left. + \int_0^1 |2h(U(t)) - h(b)| |f'(U(t)) (U(t))^2| dt \right\}$$

Using $|f'|$ is harmonically-convex in (2.7)

$$(2.8) \quad \left| [h(b) - 2h(a)] \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \int_a^b f(x)h'(x)dx \right|$$

$$\leq \frac{b-a}{4ab} \left\{ \int_0^1 |2h(L(t)) - h(b)| \{t|f'(H)| + (1-t)|f'(a)|\} (L(t))^2 dt \right.$$

$$\left. + \int_0^1 |2h(U(t)) - h(b)| \{t|f'(H)| + (1-t)|f'(b)|\} (U(t))^2 dt \right\},$$

by (2.8) and Lemma 2, this proof is complete. \square

Corollary 1. Let $h(t) = \int_{1/t}^{1/a} \left[(x - \frac{1}{b})^{\alpha-1} + (\frac{1}{a} - x)^{\alpha-1} \right] g \circ \varphi(x) dx$ for all $1/t \in$

$[\frac{1}{b}, \frac{1}{a}]$, $\alpha > 0$ and $g : [a, b] \rightarrow [0, \infty)$ be continuous positive mapping and symmetric to $\frac{2ab}{a+b}$ in Theorem 7, we obtain:

$$(2.9) \quad \left| \left(\frac{f(a) + f(b)}{2} \right) \left[J_{1/b+}^\alpha g \circ \varphi(1/a) + J_{1/a-}^\alpha g \circ \varphi(1/b) \right] - \left[J_{1/b+}^\alpha (fg \circ \varphi)(1/a) + J_{1/a-}^\alpha (fg \circ \varphi)(1/b) \right] \right|$$

$$\leq \frac{(b-a)^{\alpha+1} \|g\|_\infty}{2^{\alpha+1} (ab)^{\alpha+1} \Gamma(\alpha+1)} [C_1(\alpha) |f'(a)| + C_2(\alpha) |f'(H)| + C_3(\alpha) |f'(b)|]$$

where

$$C_1(\alpha) = \int_0^1 (1-t) [(1+t)^\alpha - (1-t)^\alpha] (L(t))^2 dt$$

$$C_2(\alpha) = \int_0^1 t [(1+t)^\alpha - (1-t)^\alpha] [(L(t))^2 + (U(t))^2] dt$$

$$C_3(\alpha) = \int_0^1 (1-t) [(1+t)^\alpha - (1-t)^\alpha] (L(t))^2 dt$$

Specially in (2.9) and using Lemma 1, for $0 < \alpha \leq 1$ we have:

$$\begin{aligned}
(2.10) \quad & \left| \left(\frac{f(a) + f(b)}{2} \right) \left[J_{1/b+}^\alpha g \circ \varphi(1/a) + J_{1/a-}^\alpha g \circ \varphi(1/b) \right] - \left[J_{1/b+}^\alpha (fg \circ \varphi)(1/a) + J_{1/a-}^\alpha (fg \circ \varphi)(1/b) \right] \right| \\
& \leq \frac{(b-a)^{\alpha+1} \|g\|_\infty}{2(ab)^{\alpha+1} \Gamma(\alpha+1)} [C_1(\alpha) |f'(a)| + C_2(\alpha) |f'(H)| + C_3(\alpha) |f'(b)|]
\end{aligned}$$

where

$$C_1(\alpha) = \int_0^1 (1-t) t^\alpha (L(t))^2 dt$$

$$C_2(\alpha) = \int_0^1 t^{\alpha+1} [(L(t))^2 + (U(t))^2] dt$$

$$C_3(\alpha) = \int_0^1 (1-t) t^\alpha (U(t))^2 dt$$

Proof. By left side of inequality (2.8) in Teorem 7, when we write $h(t) = \int_{1/t}^{1/a} \left[\left(x - \frac{1}{b}\right)^{\alpha-1} + \left(\frac{1}{a} - x\right)^{\alpha-1} \right] g \circ \varphi(x) dx$ for all $x \in [1/b, 1/a]$ and $\varphi(x) = 1/x$, we have

$$\left| \begin{aligned} & \Gamma(\alpha) \left(\frac{f(a)+f(b)}{2} \right) \left[J_{1/b+}^\alpha g \circ \varphi(1/a) + J_{1/a-}^\alpha g \circ \varphi(1/b) \right] \\ & - \Gamma(\alpha) \left[J_{1/b+}^\alpha (fg \circ \varphi)(1/a) + J_{1/a-}^\alpha (fg \circ \varphi)(1/b) \right] \end{aligned} \right|$$

On the other hand, right side of inequality (2.8)

$$\begin{aligned}
(2.11) \quad & \leq \frac{b-a}{4ab} \left\{ \int_0^1 \left| \begin{aligned} & 2 \int_{1/L(t)}^{1/a} \left[\left(x - \frac{1}{b}\right)^{\alpha-1} + \left(\frac{1}{a} - x\right)^{\alpha-1} \right] g \circ \varphi(x) dx \\ & - \int_{1/b}^{1/a} \left[\left(x - \frac{1}{b}\right)^{\alpha-1} + \left(\frac{1}{a} - x\right)^{\alpha-1} \right] g \circ \varphi(x) dx \end{aligned} \right| \{t |f'(H)| + (1-t) |f'(a)|\} (L(t))^2 dt \right. \\
& \left. + \int_0^1 \left| \begin{aligned} & 2 \int_{1/U(t)}^{1/a} \left[\left(x - \frac{1}{b}\right)^{\alpha-1} + \left(\frac{1}{a} - x\right)^{\alpha-1} \right] g \circ \varphi(x) dx \\ & - \int_{1/b}^{1/a} \left[\left(x - \frac{1}{b}\right)^{\alpha-1} + \left(\frac{1}{a} - x\right)^{\alpha-1} \right] g \circ \varphi(x) dx \end{aligned} \right| \{t |f'(H)| + (1-t) |f'(b)|\} (U(t))^2 dt \right\}
\end{aligned}$$

Since $g(x)$ is symmetric to $x = \frac{2ab}{a+b}$, we have

$$(2.12) \quad \left| 2 \int_{1/L(t)}^{1/a} \left[\left(x - \frac{1}{b}\right)^{\alpha-1} + \left(\frac{1}{a} - x\right)^{\alpha-1} \right] g \circ \varphi(x) dx - \int_{1/b}^{1/a} \left[\left(x - \frac{1}{b}\right)^{\alpha-1} + \left(\frac{1}{a} - x\right)^{\alpha-1} \right] (g \circ \varphi)(x) dx \right|$$

$$= \left| \int_{1/U(t)}^{1/L(t)} \left[\left(x - \frac{1}{b} \right)^{\alpha-1} + \left(\frac{1}{a} - x \right)^{\alpha-1} \right] (g \circ \varphi)(x) dx \right|$$

and

$$\begin{aligned} (2.13) \quad & \left| 2 \int_{1/U(t)}^{1/a} \left[\left(x - \frac{1}{b} \right)^{\alpha-1} + \left(\frac{1}{a} - x \right)^{\alpha-1} \right] g \circ \varphi(x) dx - \int_{1/b}^{1/a} \left[\left(x - \frac{1}{b} \right)^{\alpha-1} + \left(\frac{1}{a} - x \right)^{\alpha-1} \right] (g \circ \varphi)(x) dx \right| \\ &= \left| \int_{1/U(t)}^{1/L(t)} \left[\left(x - \frac{1}{b} \right)^{\alpha-1} + \left(\frac{1}{a} - x \right)^{\alpha-1} \right] (g \circ \varphi)(x) dx \right| \end{aligned}$$

for all $t \in [0, 1]$.

By (2.11)- (2.13), we have

$$\begin{aligned} (2.14) \quad & \left| \left(\frac{f(a) + f(b)}{2} \right) \left[J_{1/b+}^{\alpha} g \circ \varphi(1/a) + J_{1/a-}^{\alpha} g \circ \varphi(1/b) \right] - \left[J_{1/b+}^{\alpha} (fg \circ \varphi)(1/a) + J_{1/a-}^{\alpha} (fg \circ \varphi)(1/b) \right] \right| \\ & \leq \frac{b-a}{4ab\Gamma(\alpha)} \left\{ \int_0^1 \left| \int_{1/U(t)}^{1/L(t)} \left(x - \frac{1}{b} \right)^{\alpha-1} + \left(\frac{1}{a} - x \right)^{\alpha-1} \right| g \circ \varphi(x) dx \right| \{t|f'(H)| + (1-t)|f'(a)|\} (L(t))^2 dt \\ & \quad + \int_0^1 \left| \int_{1/U(t)}^{1/L(t)} \left[\left(x - \frac{1}{b} \right)^{\alpha-1} + \left(\frac{1}{a} - x \right)^{\alpha-1} \right] g \circ \varphi(x) dx \right| \{t|f'(H)| + (1-t)|f'(b)|\} (U(t))^2 dt \right\} \\ & \leq \frac{(b-a)\|g\|_{\infty}}{4ab\Gamma(\alpha)} \left\{ \int_0^1 \left[\int_{1/U(t)}^{1/L(t)} \left| \left(x - \frac{1}{b} \right)^{\alpha-1} + \left(\frac{1}{a} - x \right)^{\alpha-1} \right| dx \right] \{t|f'(H)| + (1-t)|f'(a)|\} (L(t))^2 dt \right. \\ & \quad \left. + \int_0^1 \left[\int_{1/U(t)}^{1/L(t)} \left| \left(x - \frac{1}{b} \right)^{\alpha-1} + \left(\frac{1}{a} - x \right)^{\alpha-1} \right| dx \right] \{t|f'(H)| + (1-t)|f'(b)|\} (U(t))^2 dt \right\}. \end{aligned}$$

In the last inequality,

$$\begin{aligned} (2.15) \quad & \int_{1/U(t)}^{1/L(t)} \left| \left(x - \frac{1}{b} \right)^{\alpha-1} + \left(\frac{1}{a} - x \right)^{\alpha-1} \right| dx = \int_{1/U(t)}^{1/L(t)} \left(x - \frac{1}{b} \right)^{\alpha-1} dx + \int_{1/U(t)}^{1/L(t)} \left(\frac{1}{a} - x \right)^{\alpha-1} dx \\ &= \frac{2^{1-\alpha}}{\alpha} \left(\frac{b-a}{ab} \right)^{\alpha} \{ (1+t)^{\alpha} - (1-t)^{\alpha} \}. \end{aligned}$$

By Lemma 1, we have

$$\begin{aligned} \int_{1/U(t)}^{1/L(t)} \left| \left(x - \frac{1}{b} \right)^{\alpha-1} + \left(\frac{1}{a} - x \right)^{\alpha-1} \right| dx &= \int_{1/U(t)}^{1/L(t)} \left(x - \frac{1}{b} \right)^{\alpha-1} dx + \int_{1/U(t)}^{1/L(t)} \left(\frac{1}{a} - x \right)^{\alpha-1} dx \\ &\leq \frac{2}{\alpha} \left(\frac{b-a}{ab} \right)^{\alpha} t^{\alpha} \end{aligned}$$

A combination of (2.14) and (2.15), we have (2.9). This complete is proof. \square

Corollary 2. *In Corollary 1,*

(1) *If $\alpha = 1$ is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for harmonically-convex function which is related the left-hand side of (2.10):*

$$\begin{aligned} (2.16) \quad & \left| \left[\frac{f(a) + f(b)}{2} \right] \int_a^b \frac{g(x)}{x^2} dx - \int_a^b f(x) \frac{g(x)}{x^2} dx \right| \leq \\ & \frac{(b-a)^2}{4(ab)^2} \|g\|_{\infty} [C_1(1) |f'(a)| + C_2(1) |f'(H)| + C_3(1) |f'(b)|] \end{aligned}$$

where for $a, b, H > 0$, we have

$$\begin{aligned} C_1(1) &= \int_0^1 (1-t)t(L(t))^2 dt \\ C_2(1) &= \int_0^1 t^2 [(L(t))^2 + (U(t))^2] dt \\ C_3(1) &= \int_0^1 (1-t)t(U(t))^2 dt \end{aligned}$$

(2) *If $g(x) = 1$ is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for harmonically-convex function which is related the left-hand side of (2.9):*

$$\begin{aligned} (2.17) \quad & \left| \left(\frac{f(a) + f(b)}{2} \right) - \frac{(ab)^{\alpha} \Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{1/b+}^{\alpha} (f \circ \varphi)(1/a) + J_{1/a-}^{\alpha} (f \circ \varphi)(1/b) \right] \right| \\ & \leq \frac{(b-a)}{2^{\alpha+2} ab} [C_1(\alpha) |f'(a)| + C_2(\alpha) |f'(H)| + C_3(\alpha) |f'(b)|]. \end{aligned}$$

(3) *If $g(x) = 1$ and $\alpha = 1$ is in corollary, we obtain following Hermite-Hadamard-Fejer Type inequality for harmonically-convex function which is related the left-hand side of (2.10):*

$$(2.18) \quad \left| \left(\frac{f(a) + f(b)}{2} \right) - \frac{ab}{(b-a)} \int_a^b \frac{f(x)}{x^2} dx \right|$$

$$\leq \frac{(b-a)}{4(ab)} [C_1(1) |f'(a)| + C_2(1) |f'(H)| + C_3(1) |f'(b)|].$$

Theorem 7. Let $f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be differentiable mapping I° , where $a, b \in I$ with $a < b$. If the mapping $|f'|^q$ is harmonically-convex on $[a, b]$, then the following inequality holds:

$$(2.19) \quad \left| [h(b) - 2h(a)] \frac{f(a)}{2} + h(b) \frac{f(b)}{2} - \int_a^b f(x) h'(x) dx \right| \\ \leq \frac{b-a}{4ab} \left\{ \left(\int_0^1 |2h(L(t)) - h(b)| dt \right)^{1-\frac{1}{q}} \times \left(\int_0^1 \frac{(|2h(L(t)) - h(b)| dt)}{t(L(t))^{2q} |f'(a)|^q + (1-t)(L(t))^{2q} |f'(H)|^q} dt \right)^{\frac{1}{q}} \right. \\ \left. + \left(\int_0^1 |2h(U(t)) - h(b)| dt \right)^{1-\frac{1}{q}} \times \left(\int_0^1 \frac{(|2h(U(t)) - h(b)| dt)}{t(U(t))^{2q} |f'(b)|^q + (1-t)(U(t))^{2q} |f'(H)|^q} dt \right)^{\frac{1}{q}} \right\}$$

Proof. Continuing from (2.7) in Theorem 7, we use Hölder Inequality and we use that $|f'|^q$ is harmonically-convex. Thus this proof is complete. \square

Corollary 3. Let $h(t) = \int_{1/t}^{1/a} \left[\left(x - \frac{1}{b}\right)^{\alpha-1} + \left(\frac{1}{a} - x\right)^{\alpha-1} \right] (g \circ \varphi)(x) dx$ for all $t \in [a, b]$ and $g : [a, b] \rightarrow [0, \infty)$ be continuous positive mapping and symmetric to $\frac{2ab}{a+b}$ in Teorem 8, we obtain:

$$(2.20) \quad \left| \left(\frac{f(a) + f(b)}{2} \right) \left[J_{1/b+}^\alpha (g \circ \varphi)(1/a) + J_{1/a-}^\alpha (g \circ \varphi)(1/b) \right] - \left[J_{1/b+}^\alpha (fg \circ \varphi)(1/a) + J_{1/a-}^\alpha (fg \circ \varphi)(1/b) \right] \right|$$

$$\leq \frac{(b-a)^{\alpha+1} \|g\|_\infty}{2^{\alpha+1} (ab)^{\alpha+1} \Gamma(\alpha+1)} \left(\frac{2^2(2^\alpha-1)}{\alpha+1} \right)^{1-\frac{1}{q}} [C_1(\alpha, q) |f'(a)|^q + C_2(\alpha, q) |f'(H)|^q + C_3(\alpha, q) |f'(b)|^q]^{\frac{1}{q}}$$

where for $q > 1$

$$C_1(\alpha, q) = \int_0^1 [(1+t)^\alpha - (1-t)^\alpha] t (L(t))^{2q} dt$$

$$C_2(\alpha, q) = \int_0^1 [(1+t)^\alpha - (1-t)^\alpha] (1-t) \left((L(t))^{2q} + (U(t))^{2q} \right) dt$$

$$C_3(\alpha, q) = \int_0^1 [(1+t)^\alpha - (1-t)^\alpha] t (U(t))^{2q} dt.$$

Proof. Continuing from (2.15) of Corollary 1 and (2.19) in Theorem 8,

$$\begin{aligned}
 (2.21) \quad & \left| \left(\frac{f(a) + f(b)}{2} \right) [J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)] - [J_{a^+}^\alpha (fg)(b) + J_{b^-}^\alpha (fg)(a)] \right| \\
 & \leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+1} \Gamma(\alpha+1)} \left\{ \left(\int_0^1 [(1+t)^\alpha - (1-t)^\alpha] dt \right)^{1-\frac{1}{q}} \times \right. \\
 & \quad \left(\int_0^1 [(1+t)^\alpha - (1-t)^\alpha] \left(t(L(t))^{2q} |f'(a)|^q + (1-t)(L(t))^{2q} |f'(H)|^q \right) dt \right)^{\frac{1}{q}} \\
 & \quad + \left(\int_0^1 [(1+t)^\alpha - (1-t)^\alpha] dt \right)^{1-\frac{1}{q}} \times \\
 & \quad \left. \left(\int_0^1 [(1+t)^\alpha - (1-t)^\alpha] \left(t(U(t))^{2q} |f'(b)|^q + (1-t)(U(t))^{2q} |f'(H)|^q \right) dt \right)^{\frac{1}{q}} \right\} \\
 & \leq \frac{(b-a)^{\alpha+1} \|g\|_\infty}{2^{\alpha+1} (ab)^{\alpha+1} \Gamma(\alpha+1)} \left(\frac{2^{\alpha+1} - 2}{\alpha+1} \right)^{1-\frac{1}{q}} \left[\left(\int_0^1 \left[\begin{aligned} & [(1+t)^\alpha - (1-t)^\alpha] \times \\ & \left[t(L(t))^{2q} |f'(a)|^q + (1-t)(L(t))^{2q} |f'(H)|^q \right] \end{aligned} \right] dt \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left(\int_0^1 \left[\begin{aligned} & [(1+t)^\alpha - (1-t)^\alpha] \times \\ & \left[t(U(t))^{2q} |f'(b)|^q + (1-t)(U(t))^{2q} |f'(H)|^q \right] \end{aligned} \right] dt \right)^{\frac{1}{q}} \right]
 \end{aligned}$$

By the power-mean inequality $(a^r + b^r) < 2^{1-r} (a+b)^r$ for $a > 0, b > 0, r < 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\begin{aligned}
 (2.22) \quad & \leq \frac{(b-a)^{\alpha+1} \|g\|_\infty}{2^{\alpha+1} (ab)^{\alpha+1} \Gamma(\alpha+1)} \left(\frac{2^2(2^\alpha - 1)}{\alpha+1} \right)^{\frac{1}{p}} \left[\int_0^1 \left(\begin{aligned} & [(1+t)^\alpha - (1-t)^\alpha] t(L(t))^{2q} |f'(a)|^q + \\ & [(1+t)^\alpha - (1-t)^\alpha] (1-t) \left(\frac{(L(t))^{2q}}{(U(t))^{2q}} \right) |f'(H)|^q \\ & + [(1+t)^\alpha - (1-t)^\alpha] t(U(t))^{2q} |f'(b)|^q \end{aligned} \right) dt \right]^{\frac{1}{q}}
 \end{aligned}$$

□

Corollary 4. When $\alpha = 1$ and $g(x) = 1$ is taken in Corollary 3, we obtain:

$$\begin{aligned}
 (2.23) \quad & \left| \left(\frac{f(a) + f(b)}{2} \right) - \frac{ab}{(b-a)} \int_a^b \frac{f(x)}{x^2} dx \right| \\
 & \leq \frac{(b-a)}{2^{2+\frac{1}{q}} (ab)} [C_1(1, q) |f'(a)|^q + C_2(1, q) |f'(H)|^q + C_3(1, q) |f'(b)|^q]^{\frac{1}{q}}
 \end{aligned}$$

This proof is complete.

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DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES,, GİRESUN UNIVERSITY, 28100, GİRESUN, TURKEY.

E-mail address: imdati@yahoo.com, imdat.iscan@giresun.edu.tr

DERELİ VOCATIONAL HIGH SCHOOL,, GİRESUN UNIVERSITY, 28100, GİRESUN, TURKEY.

E-mail address: sercanturhan28@gmail.com, sercan.turhan@giresun.edu.tr

DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES,, ORDU UNIVERSITY, 52000, ORDU, TURKEY.

E-mail address: maden55@mynet.com